LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – **APRIL 2016**

MT 1816 - REAL ANALYSIS

Date: 03-05-2016   Dept. No.   Max. : 100 Marks

Time: 01:00-04:00

Answer all the Questions. Each question carries 20 marks.

1. (a) If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on [a, b], then prove that $|f| \in \Re(\alpha)$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha.$

(OR)

(b) Prove that $\int_{\underline{a}}^{b} f d\alpha \leq \int_a^{\overline{b}} f d\alpha.$ **(5 marks)**

(c) i) Prove that $f \in \Re(\alpha)$ on [a, b] if and only if for every $\epsilon > 0$, there exists a partition **P** such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. **(10 marks)**

ii) Any monotone function f: [0, 1]→R is Riemann Integrable. Justify. **(5 marks)**

(OR)

(d) i) Suppose $f \in \Re(\alpha)$ on [a, b], $m \leq f \leq M$, ϕ is continuous on [m, M] and h(x) = ϕ(f(x)) on [a, b]. Then prove that $h \in \Re(\alpha)$ on [a, b]. **(10 marks)**

ii) State and prove the fundamental theorem of calculus. **(5 marks)**

2. (a) Let α be monotonically increasing on $[a,b]$, $f_n \varepsilon \Re(\alpha)$ on $[a,b]$, for $n = 1,2,3,...$ and suppose $f_n \to f$ uniformly on $[a,b]$. Then prove that $f \varepsilon \Re(\alpha)$ on $[a,b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.

(OR)

(b) Illustrate with an example that the limit of the integral need not be equal to the integral of the limit. **(5 marks)**

(c) State and prove the Stone-Weierstrass theorem.

(OR)

(d) If $\{f_n\}$ is a sequence of continuous functions on a set *E* and if $f_n \to f$ uniformly on *E*, then prove that *f* is continuous on *E*. **(15 marks)**

3. (a) Let $S = \{\varphi_0, \varphi_1, \varphi_2, ..\}$, where $\varphi_0(x) = \frac{1}{\sqrt{2\pi}}$, $\varphi_{2n-1}(x) = \frac{\cos nx}{\sqrt{\pi}}$ and

$\varphi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}}$, for n = 1, 2…. Prove that S is orthnormal on any interval of length 2 π.

(OR)

(b) State and prove the Bessel's Inequality and Parseval's formula. **(5 marks)**

(c) State and prove the Riemann-Lebesgue lemma and use the lemma to prove the following:

For $f \in L(-\infty, +\infty)$, $\lim_{\alpha \to \infty} \int_{-\infty}^{\infty} f(t) \frac{1-\cos\alpha t}{t} dt = \int_0^{\infty} \frac{f(t)-f(-t)}{t} dt.$ **(15 marks)**

(OR)

(d) (i) Define Dirichlet's kernel and prove that $\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(2n+1)\frac{x}{2}}{2\sin\frac{x}{2}}, x \neq 2m\pi$

(ii) If $f \in L[0,2\pi]$, f is periodic with period 2π and $\{s_n\}$ is a sequence of partial sums of Fourier series generated by f. $s_n = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), n = 1,2$ ... then

prove that $s_n(x) = \frac{2}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n(t)dt.$ **(5+10 marks)**

4. (a) If A, B$\in$L(R^n, R^m)and c is a scalar, then prove that, $\|A + B\| \leq \|A\| + \|B\|$ and $\|cA\| = |c|\|A\|$

(OR)

(b) State and prove the fixed point theorem for a complete metric space. **(5 marks)**

(c) State and prove the inverse function theorem.

(OR)

(d) State and prove the implicit function theorem. **(15 marks)**

5. (a) Define heat flow and the heat equation.

(OR)

(b) Explain rectilinear coordinate system with algebraic and geometric approach.

(5 marks)

(c) Derive the expression for Newton's Law of Cooling.

(OR)

(d) Derive the D' Alembert's wave equation for a vibrating string. **(15 marks)**

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