LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2016**

MT 2814 - COMPLEX ANALYSIS

Date: 25-04-2016 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

Answer all the questions:

1. (a) Let G be a region. Show that any analytic function $f: G \to \mathbb{C}$ such that $|f(z)| \leq |f(a)| \; \forall \; z \in G$ is constant. **(5)**

OR

(b) Define (i) Zeros of an analytic function (ii) index of a closed curve (iii) FEP homotopic (iv) Simply connected. **(5)**

(c) (i) Let f be analytic in $B(a; R)$ then prove that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$ where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\geq R$.

(ii) State and prove Morera's theorem. **(7+8)**

OR

(d) State and prove homotopic version of Cauchy's theorem. **(15)**

2. (a) Define convex function and prove that a function $f: [a, b] \to \mathbb{R}$ is convex if and only if $A = \{(x, y): a \leq x \leq b \; and \; f(x) \leq y\}$ is convex. **(5)**

OR

(b) If a set $\mathfrak{F} \subset C(G, \Omega)$ is normal then prove that the following conditions are satisfied:(i) for each z in G, $\{f(z) : f \in \mathfrak{F}\}$ has compact closure in Ω (ii) $\mathfrak{F}$ is equicontinuous at each point of G. **(5)**

(c) Let $a < b$ and let G be a vertical strip $\{x + iy: a < x < b\}$. Suppose $f: G^- \to \mathbb{C}$ is continuous and f is analytic in G. If we define $M: [a, b] \to \mathbb{R}$ by $M(x) = sup\{|f(x + iy)|: -\infty < y < \infty\}$, and $|f(z)| < B$ for all z in G, then prove that $logM(x)$ is a convex function. **(15)**

OR

(d) (i) State and prove Hadamard's three circles theorem.

(ii) Let G be a region which is not the whole plane and such that every non-vanishing analytic function on G has an analytic square root. If $a \in G$ then prove that there is an analytic function f on G such that (a) $f(a) = 0$ and $f'(a) > 0$ (b) f is one-one and (c) $f(G) = D = \{z: |z| < 1\}$.

(7+8)

3. (a) State and prove Gauss's formula. **(5)**

OR

(b) Show that $\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - log n\right]$ in the usual notation. **(5)**

(c) (i) Define elementary factor and prove that $|1 - E_p(z)| \leq |z|^{p+1}$ for $|z| \leq 1$ and $p \geq 0$.

(ii) State and prove Bohr-Mollerup theorem. **(7+8)**

OR

(d) (i) For $Re\, z > 1$ then prove that $\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt$.

(ii) If $Re\, z > 0$ then prove that $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ **(7+8)**

4. (a) Define rank, genus and order of an entire function. **(5)**

OR

(b) State and prove Poisson-Jensen Formula. **(5)**

(c) State and prove Mittag- Leffler's theorem. **(15)**

OR

(d) State and prove Hadamard Factorization theorem. **(15)**

5. (a) Define elliptic function and prove that a non constant elliptic function has equally many poles as it has zeros. **(5)**

OR

(b) Prove that $\wp'(z) = \frac{-\sigma(2z)}{(\sigma(z))^4}$ **(5)**

(c) (i) Prove that the zeros $a_1, a_2, \ldots, a_n$ and poles $b_1, b_2, \ldots, b_n$ of an elliptic function satisfy $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n (mod\ M)$.

(ii) Prove that $\wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right)^2$. **(8+7)**

OR

(d) (i) Show that $\zeta(z) = \frac{1}{z} + \sum_{w\neq 0}\left(\frac{1}{(z-w)} + \frac{z}{w^2} + \frac{1}{w}\right)$ and it is an odd function. Also show that $\zeta'(z) = -\wp(z)$.

(ii) Prove that a discrete module consists of either of zero alone, of the integral multiples nw of a single complex number $w \neq 0$ or of linear combinations $n_1 w_1 + n_2 w_2$ with integral coefficients of two numbers w_1, w_2 with non real ratio $\frac{w_2}{w_1}$. **(8+7)**

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