LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

B.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIFTHSEMESTER – APRIL 2017

MT 5509- ALGEBRAIC STRUCTURE - II

Date: 28-04-2017 Dept. No. Max. : 100 Marks
01:00-04:00

PART A

ANSWERALL THE QUESTIONS $(\mathbf{10 * 2 = 20}\, \boldsymbol{marks})$

1. If V is a vector space over F then show that $(-a)v = a(-v) = -(av)$ for $a \in F, v \in V$.
2. Express the vector (1,-2,5) as a linear combination of the vectors (1,1,1),(1,2,3) and(2,-1,1) in $\mathcal{R}^3$ where $\mathcal{R}$ is the field of real numbers.
3. Prove that the vectors (1,0,0), (1,1,0) and (1,1,1) form a basis of R^3, where R is the field of real numbers.
4. Define rank and nullity of a vector space homomorphism $T : U \to V$.
5. Let R^3 be the inner product over R under standard inner product. Find the norm of (3,0,4).
6. Let $T \in A(v)$ and $\lambda \in F$. Then prove that λ is an eigenvalue of T if and only if $\lambda I - T$ is singular.
7. Define trace of a matrix and give an example.
8. If A is any square matrix, prove that $A + A^t$ is symmetric and $A - A^t$ is a skew-symetric.
9. Find the rank of the matrix A= $\begin{pmatrix} 1 & 5 & -7 \\ 2 & 3 & 1 \end{pmatrix}$ over the field of rational numbers.
10. If $T \in A(V)$ is Hermitian, then prove that all its eigenvalues are real.

PART B

ANSWERANY FIVE QUESTIONS $(\mathbf{5 * 8 = 40}\, \boldsymbol{marks})$

11. Prove that the union of two subspaces of a vector spaces V over F is a subspace of V if and only if one is contained in the other.
12. If S and T are subsets of a vector space V over F, then prove the following:
    i) S is subspace of V if and only if $L(S) = S$.
    ii) $S \subseteq T$ implies that $L(S) \subseteq L(T)$.
13. Let V be a vector space and suppose that one basis has n elements and another basis has m elements .Then prove that $m = n$.
14. If A and B are subspaces of a vector space V over F, prove that $(A + B)/B \cong A/A \cap B$.
15. Apply the Gram-Schmidt orthonormalization process to obtain an orthonormal basis for the subspace of R^4 generated by the vectors (1,1,0,1), (1, −2,0,0), *and* (1,0, −1,2).
16. If $\lambda \in F$ is an eigenvalue of $T \in A(V)$, then prove that for any polynomial $f(x) \in F[x]$, $f(\lambda)$ is an eigenvalue of $f(T)$.
17. Show that any square matrix A can be expressed uniquely as the sum of a symmetric matrix and a skew-symmetric matrix.
18. Investigate for what values of λ, μ the system of equations $x_1 + x_2 + x_3 = 6, x_1 + 2x_2 + 3x_3 = 10, x_1 + 2x_2 + \lambda x_3 = \mu$ over the rational field has a) no solution b) a unique solution c) an infinite number of solutions.

PART C

ANSWERANY TWO QUESTIONS $(\mathbf{2*20=40}\ \boldsymbol{marks})$

19. a) Prove that the vector space V over F is a direct sum of two of its subspaces W_1 and W_2 if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = (0)$.

b) If V is a vector space of finite dimension and is the direct sum of its subspaces of U and W then prove that $\dim V = \dim U + \dim W$. (10+10)

20. a) If V is a vector space of dimension n then prove that

i) any $n+1$ vectors in V are linearly dependent.

ii) Any set of n linearly independent vectors V is a basis of V.

b) If U and V are vector spaces over F, and if T is a homomorphism of U onto V with kernel W, then prove that $U/W \cong V$.

21. a) If u, v are any two vectors in V then prove that $||u+v|| \leq ||u|| + ||v||$.

b) Prove that $T \in A(V)$ is singular if and only if there exists an element $v \neq 0$ in V such that $T(V) = 0$.

22. a) If $A, B \in F_n$ and if $\lambda \in F$, then prove that

i) $(\lambda A)^t = \lambda A^t$

ii) $(A^t)^t = A$

iii) $(A+B)^t = A^t + B^t$

iv) $(AB)^t = B^t A^t$

b) Prove that the eigenvalues of a unitary transformations are all of its absolute value 1.

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