LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2018**

16PMT4MC01– FUNCTIONAL ANALYSIS

Date: 18-04-2018 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

Answer ALL the questions:

1. (a) Define (i) Ray (ii) Hyperplane (iii) Algebraic Dual. **(5)**

(OR)

(b) If X is a vector space, Y and Z are subspaces of X and is complementary to Z, then prove that every element of X/Y contains exactly one element of Z. **(5)**

(c) (i) Prove that every vector space Xcontains a set of linearly independent elements which generates X.

(ii) Prove that if $f \in X^*$, then $Z(f)$ has deficiency 0 or 1 in X. Conversely, if Z is a subspace of X of deficiency 0 or 1, then there is an $f \in X^*$ such that $Z = Z(f)$. **(7+8)**

(OR)

(d) Let X be a real vector space, let Y be a subspace of X and p be a real valued function on X such that $p(x+y) \leq p(x) + p(y)$ and $p(ax) = ap(x)$ for all $x, y \in X$, for $a \geq 0$. If f is a linear functional on Y and $f(x) \leq p(x)$ for all $x \in Y$, prove that there is a linear functional F on X such that $F(x) = f(x)$ for all $x \in Y$and $F(x) \leq p(x)$ for all $x \in X$. **(15)**

2. (a) Let $B(X,Y)$ be the set of all bounded linear transformation of X into Y. Prove that $B(X,Y)$ is a normed vector space which is Banach space if Y is a Banach space. **(5)**

(OR)

(b) State and prove F.Rieszlemma. **(5)**

(c) (i) State and prove Hahn Banach Theorem for a real normed linear space.

(ii) Let Xbe a real normed linear space. Then prove that for any $x \neq 0$ in X there is an $x' \in X'$such that$x'(x) = \|x\|$and$\|x'\| = 1$. **(10+5)**

(OR)

(d) State and prove Hahn Banach Theorem for a complex normed linear space. **(15)**

3. (a) Let X and Y be Banach spaces and let T be a linear transformation of X into Y. Prove that if the graph of T is closed then T is bounded. **(5)**

(OR)

(b) Define a dual space. Let X be a normed vector space and let X' be the dual space of X. If X' is separable then prove that X is separable. **(5)**

(c)(i) Mention any two properties of the projection. If P is a projection on a Banach space X and if M and N are its range and null space respectively then prove that M and N are closed linear subspaces of X such that $X = M \oplus N$.

(ii) If M is a direct sum of X and N is a closed subspace with $X = M \oplus N$ and with unique representation $x = y + z$ where $y \in M$, $z \in N$ then prove that P is a projection where $Px = y$.

(iii) Show that a Banach space cannot have a countably infinite basis. **(7+6+3)**

(OR)

(d) State and prove open mapping theorem. **(15)**

4. (a) Let M be a non empty subset of a Hilbert space H. Then prove that $M^{\perp}$ is a closed linear subspace of H.

(5)

(OR)

(b) Prove that the self adjoint operators in $B(H)$ form a closed real linear subspace of $B(H)$ and therefore a real Banach space. **(5)**

(c) State and prove Riesz Fischer theorem. **(15)**

(OR)

(d) If $P_1, P_2, \dots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \dots, M_n$ of a Hilbert space H, then prove that $P = P_1 + P_2 + \cdots + P_n$ is a projection if and only if P_i's are pairwiseorthogonal. Also, show that P is a projection on $M = M_1 + M_2 + \cdots + M_n$.

(15)

5. (a) Define a Banach algebra A and prove that the set of regular elements in A is open. **(5)**

(OR)

(b) Let A be Banach algebra. Show that the inverse of the regular element $x \in A$ is $x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$.

(5)

(c) (i) Define topological divisor of zero in Banach Algebra A. Let Z denote the set of all topological divisors of zero in A. Then prove that every zero divisor in A is a topological divisor in A. Also prove that G is an open set and therefore S is closed set. **(2+2+3)**

(ii) Prove that the mapping $f : G \to G$ given by $f(x) = x^{-1}$ is continuous and is a homeomorphism.

(8)

(OR)

(d) State and prove the Spectral theorem. **(15)**

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