LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc., DEGREE EXAMINATION - **MATHEMATICS**

SECOND SEMESTER – **APRIL 2022**

PMT 2501 – ALGEBRA

Date: 15-06-2022 Dept. No. Max. : 100 Marks

Time : 09:00 A.M. – 12:00 NOON

Answer **ALL** the Questions.

1. a) If O (G) = p^2 where p is a prime number, then show that G is abelian.

(OR) **(5)**

b) Prove that a group of order 72 is not simple.

c) If p is a prime number such that p^α divides order of G then prove that G has a subgroup of order p^α.

(OR) **(15)**

d) State and prove Cauchy's theorem and prove that the number of p-sylow subgroups in G is of the form 1 + kp.

2. a) For the given two polynomials $f(x), g(x) \neq 0$ in $F[x]$ prove that there exists two polynomials t(x), r(x) in F[x] such that $f(x)=t(x)\,g(x)+r(x)$ where $r(x)=0$ (or) deg $r(x) < \deg g(x)$.

(OR) **(5)**

b) If $f(x)$ and $g(x)$ are primitive polynomials prove that $f(x)\ g(x)$ is also a primitive polynomial.

c) (i) If the primitive polynomial $f(x)$ can be factored as the product of two polynomials having rational coefficients prove that it can be factored as the product of two polynomials having integer coefficients. **(7)**

(ii) If $f(x)$ and $g(x)$ are two nonzero polynomials prove that $\deg\left(f(x)g(x)\right) = \deg f(x) + \deg\ g(x)$. **(8)**

(OR)

d) (i) State and prove Eisenstein Criterion. **(8)**

(ii) State and prove Gauss Lemma. **(7)**

3. a) Prove that the elements in K which are algebraic over F form a subfield of K.

(OR) **(5)**

b) Let F be a field of rational numbers and let $f(x)=x^3-2$. Find the degree of the splitting field E over F.

c) Prove that the element $a \in K$ is said to be algebraic over F iff F(a) is a finite extension over F. **(15)**

(OR)

d) (i) If L is the finite extension of K and K is the finite extension of F prove that L is the finite extension of F. **(10)**

(ii) If L is the finite extension of F and K is the subfield of L which contains F prove that [K : F] divides [L : F]. **(5)**

4. a) Prove that a polynomial of degree n over the field F can have atmost 'n' roots in any extension field.

(OR) **(5)**

b) Prove that K is a normal extension of F iff K is a splitting field of some polynomial over F.

c) State and prove fundamental theorem of Galois Theory.

(OR) **(15)**

d) Let K be a normal extension of F and let H be a subgroup of G(K,F), $K_H = \{x \in K / \sigma(x) = x \,\forall \sigma \in H\}$ is a fixed field of H prove that i) $[K : K_H] = O(H)$, ii) $H = G(K, K_H)$, in particular, $H = G(K,F)$, $[K : F] = O(G(K, F))$.

5. a) Let G be a finite abelian group such that $x^n = (e)$ is satisfied by atmost n elements of G for every n prove that G is a cyclic group.

(OR) **(5)**

b) Prove that for every prime number p and every integer m, there exists a field having p^m elements.

c) Prove that any finite division ring is necessarily a commutative field.

(OR) **(15)**

d) Prove that S_n is not solvable for $n \geq 5$ and verify S_3 is solvable.

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