LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2022**

PMT 2502 – MEASURE THEORY AND INTEGRATION

Date: 20-06-2022 Dept. No. Max. : 100 Marks

Time: 09:00 AM - 12:00 NOON

Answer ALL Questions:

1. a) For $k > 0$ and $A \subseteq R$, $kA = \{x: k^{-1}x \in A\}$, show that $m^*(kA) = km^*(A)$. **(5 Marks)**

OR

b) Let c be any real number and f, g be real-valued measurable functions defined on the same measurable set E. Then prove that functions $f + c, cf$ are measurable. **(5 Marks)**

c) If $m^*(E) < \infty$, then prove that E is measurable if and only if for all $\epsilon > 0$, there exists disjoint finite intervals $I_1, I_2 \dots I_n$ such that $m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon$. **(15 Marks)**

OR

d) Construct a non-measurable set. **(15 Marks)**

2. a) If f is an integrable function, then prove that af is integrable and $\int af\, dx = a \int f dx$. **(5 Marks)**

OR

b) State and prove Lebesgue Monotone Convergence theorem. **(5 Marks)**

c) State and prove Lebesgue Dominated convergence theorem. Use it to evaluate the integral $\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x}{1+n^2x^2} dx, x \in [0,1], n \geq 1$. **(15 Marks)**

OR

d) Prove that Riemann integrability implies Lebesgue integrability. Is the converse true? Justify. **(15 Marks)**

3. a) If μ^* is the outer measure on $\mathcal{H}(\mathcal{R})$ defined by μ on $\mathcal{R}$, then establish that the class of μ^*-measurable sets S^* contains the $\sigma - ring\ S(\mathcal{R})$ generated by $\mathcal{R}$. **(5 Marks)**

OR

b) If $\mu^*(X) < \infty$. Prove that $E \subseteq X$ is μ^*-measurable if and only if $\mu^*(X) = \mu^*(E) + \mu^*(E^c)$. **(5 Marks)**

c) Let μ^* be the outer measure on $\mathcal{H}(\mathcal{R})$ and $\mathcal{S}^*$ denotes the class of μ^*-measurable sets. Prove that $\mathcal{S}^*$ is σ-ring and μ^* restricted to $\mathcal{S}^*$ is a complete measure. **(15 Marks)**

OR

d) Prove that $\bar{\mathcal{S}}$ is a $\sigma - ring$ where μ is a measure defined on a $\sigma - ring\ \mathcal{S}$ and $\bar{\mathcal{S}} = \{E \Delta N : E \in \mathcal{S}$ and N is contained in some set in $\mathcal{S}\ with$ zero measure$\}$. Also, prove that the set function $\bar{\mu}$ defined by $\bar{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\bar{\mathcal{S}}$. **(15 Marks)**

4. a) If ψ be a convex function defined on (a, b) with $a < s < t < u < b$, then $\psi(s,t) \leq \psi(s,u) \leq \psi(t,u)$. Justify this statement. **(5 Marks)**

OR

b) When do you say that a sequence of measurable functions converges to a measurable function almost uniformly. If a sequence of measurable functions converges almost uniformly, then will it imply that the sequence converges in measure. **(5 Marks)**

c) (i) State and prove Holder's inequality. When does the equality occur? **(9 Marks)**

(ii) If $p \geq 1$ and $f, g \in L^p(\mu)$, then demonstrate $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. **(6 Marks)**

OR

d) (i) State and prove Completeness theorem for convergence in measure. **(9 Marks)**

(ii) Let $\{f_n\}$ be a sequence of non negative measurable functions and let f be a measurable function such that $f_n \overset{m}{\rightarrow} f$, then prove that $\int f d\mu \leq \lim\inf \int f_n d\mu$. **(6 Marks)**

5. a) Show that if ν is a signed measure, $|\nu(E)| < \infty$ and $F \subseteq E$, then $|\nu(F)| < \infty$. **(5 Marks)**

OR

b) Let μ, λ, ν be σ-finite signed measures on $[\![X, S]\!]$ such that $\nu \ll \mu$, $\mu \ll \lambda$, then show that $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} [\lambda]$. **(5 Marks)**

c) (i) For a signed measure ν defined on a measurable space $[\![X, S]\!]$, prove that there exists a positive set A and a negative set B such that $A \cup B = X$ and $A \cap B = \emptyset$. **(6 Marks)**

(ii) State and prove Lebesgue decomposition theorem. **(9 Marks)**

OR

d) State and prove Radon-Nikodym theorem. **(15 Marks)**

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