LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2023**

PMT2MC02 – REAL ANALYSIS-II

Date: 02-05-2023 Dept. No. Max. : 100 Marks

Time: 01:00 PM - 04:00 PM

	SECTION A – K1 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
1.	**Answer the following**
a)	What do you say about the existence of the simultaneous limit $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}, x^2+y^6 \neq 0$.
b)	Find the outer measure of a finite set.
c)	When do you say that a complex-valued function defined on a measure space is Lebesgue measurable.
d)	Let $X = (0,1]$ and define $f: X \to R$ by $f(x) = x^{-1/3}$, show that $f \in L^1(X)$ but $f \notin L^3(X)$.
e)	Give an example showing that Hahn decomposition is not unique.
	SECTION A – K2 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
2.	**Multiple Choice Questions**
a)	For what value of k, the function $f(x,y) = \begin{cases} x^2+2y, & (x,y) \neq (1,2) \\ k, & (x,y) = (1,2) \end{cases}$ is continuous at the point $(1,2)$? (i)1 (ii) 2 (iii) 3 (iv) 5
b)	For a class of subsets of an arbitrary space X, the smallest σ-algebra of subsets of X is (i) $\{\emptyset\}$ (ii) $\{X\}$ (iii) $\{\emptyset, X\}$ (iv) $\mathcal{P}(X)$(power set of X)
c)	If f is an integrable function, then (i) $\lvert\int f dx\rvert \geq \int \lvert f\rvert dx$. (ii) f is finite-valued a.e. (iii) $\int f dx = \lim_{a\to\infty} \lim_{b\to-\infty} \int_b^a f dx \neq \lim_{b\to-\infty} \lim_{a\to\infty} \int_b^a f dx$. (iv) $\int_E f dx = \int \chi_E dx$.
d)	A measure μ defined on a ring $\mathcal{R}$ is a σ-finite measure, if for every set $E \in \mathcal{R}$, (i) $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n) = 0$ for each n. (ii) $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n)$ is finite for at least one n. (iii) $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n)$ is finite for each n. (iv) $E \subset \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n) < \infty$ for each n.
e)	For a signed measure ν defined on $[\![X, S]\!]$, which of the following is true? (i) Values of ν need not be extended real numbers. (ii) ν takes both the values $\infty, -\infty$. (iii) ν is a measure. (iv) ν is countably additive for any sequence $\{E_i\}$ of disjoint measurable sets.
	SECTION B – K3 (CO2)

	Answer any THREE of the following **(3 x 10 = 30)**
3.	For a map f defined on an open set $E \subset R^n$ into R^m, show that $f \in C'(E)$ if and only if the partial derivatives $D_j f_i$ exist and continuous on E for $1 \le i \le m, 1 \le j \le n$.
4.	Establish the four equivalent definitions of a Lebesgue measurable function of an extended real-valued function defined on a measurable set and prove their equivalence.
5.	Calculate the value of $\int_0^1 \frac{x^{1/3}}{1-x} log \frac{1}{x} dx$.
6.	Let $\{f_n\}$ be a sequence of measurable functions which is fundamental in measure. Determine the measurable function f such that $f_n \overset{m}{\rightarrow} f$.
7.	If $f \in L^1(\mu \times \nu)$, then show that $f_x \in L^1(\nu)$ for almost all $x \in X$, $f^y \in L^1(\mu)$ for almost all $y \in Y$, the functions φ, ψ defined by $\varphi(x) = \int_Y f_x \, d\nu$, $\psi(y) = \int_X f^y \, d\mu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively and $\int_X \varphi \, d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y \psi d\nu$.

SECTION C – K4 (CO3)

	Answer any TWO of the following **(2 x 12.5 = 25)**
8.	Construct a non-measurable set.
9.	For a bounded function f defined on a closed finite interval, if $R \int_a^b f(x)dx$ exists then examine whether the Lebesgue integral of $f(x)$ exists but not conversely.
10.	Let μ be a measure on a σ-ring $\mathcal{S}$ and $\bar{\mathcal{S}}$ class of sets of the form $E \,\Delta\, N$ for any sets E, N such that $E \in \mathcal{S}$ while N is contained in some set in $\mathcal{S}$ of zero measure. Establish that $\bar{\mathcal{S}}$ is σ-ring and the set function $\bar{\mu}$ defined by $\bar{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\bar{\mathcal{S}}$.
11.	Prove that a signed measure defined on a measurable space $[\![X, \mathcal{S}]\!]$ can be decomposed into difference of two measures and such decomposition is unique.

SECTION D – K5 (CO4)

	Answer any ONE of the following **(1 x 15 = 15)**
12.	Evaluate the Hausdorff dimension of the Cantor-like set P_ξ.
13.	Let ν be a signed measure on a measurable space $[\![X, \mathcal{S}]\!]$ and $\nu(E) > 0 \; for \; E \in \mathcal{S}$. Verify whether there exists a positive set A with respect to ν such that $A \subseteq E$ and $\nu(A) > 0$.

SECTION E – K6 (CO5)

	Answer any ONE of the following **(1 x 20 = 20)**
14.	Let f be a C'-mapping of an open set $E \subset R^{n+m}$ into R^n such that $f(a, b) = 0$ for a point $(a, b) \in E$ and $A = f'(a, b)$. Assume that A_x is invertible. Then there exist open sets $U \subset R^{n+m}$ and $W \subset R^m$, with $(a, b) \in U$ and $b \in W$, such that: (i) To every $y \in W$ there is a unique x such that $(x, y) \in U$ and $f(x, y) = 0$. (ii) If x is defined by $g(y)$, then g is a C'-mapping of W into R^n, $g(b) = a, f(g(y), y) = 0 \;\; for \; y \in W$ and $g'(b) = -(A_x)^{-1} A_y$. Justify the statement. Use the statement to conclude that there exists a continuously differentiable mapping from the

	neighbourhood W of $(1,1)$ into R^2 such that $g(b) = a$ and find $g'(b)$, for a continuous differentiable mapping f of an open set $E \subset R^5$ into R^3 defined by $f_1(x_1, x_2, x_3, y_1, y_2) = y_1x_1 + y_1x_2 + y_2x_3 - 3,$ $f_2(x_1, x_2, x_3, y_1, y_2) = y_1x_1^2 + y_1x_2^2 + y_2x_3^2 - 5$ $f_3(x_1, x_2, x_3, y_1, y_2) = y_1x_1^3 + y_1x_2^3 + y_2x_3^3 - 9$ and $a = (0,1,2),\ b = (1,1)$.
15.	Validate the statement: For a sequence $\{f_n\}_{n\geq 1}$, of non-negative measurable functions, $liminf \int f_n dx \geq \int liminf f_n dx$. Give your comments with an example that non-negative condition of all f_n is necessary for the above statement.

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