LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – NOVEMBER 2007

MT 1805 - REAL ANALYSIS

AB 20

Date : 27/10/2007 Dept. No. Max. : 100 Marks

Time : 1:00 - 4:00

a)(1) If f is a monotonically increasing function and α is a continuous function on [a,b] then prove that $f \in \Re(\alpha)$ on [a,b].

OR

(2) Prove: $f \in \Re(\alpha)$ on [a,b] if and only if given $\in > 0$, there exists a partition P of [a,b] such that $U(P, f, \alpha) - L(P, f, \alpha) < \in$. (5)

b) (1) Let f: [a.b]→**R** be a bounded function and α be a monotonically increasing function on [a,b] then prove that $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$. (8)

(2) Suppose $f \in R$ on [a,b]. If there is a differentiable function F on [a,b] such that $F'(x) = f(x)$, $x \in [a,b]$ then prove that $\int_a^b f(x)dx = F(b) - F(a)$. (7)

OR

(3) Let $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on [a,b] then prove that fg and $|f| \in \Re(\alpha)$ on [a,b]. (7)

(4) Let $f \in \Re(\alpha)$ on [a,b] and $m \leq f \leq M$. Suppose that ϕ is continuous on [m,M]. Define $h(x) = \phi(f(x))$, $\forall x \in [a,b]$ then prove that $h \in \Re(\alpha)$ on [a,b]. (8)

II.

a) Prove the following results:

(i) If $A, B \in L(R^n, R^m)$ then $\|A+B\| \leq \|A\| + \|B\|$ and

(ii) If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^n)$ then $\|BA\| \leq \|B\|\|A\|$.

OR

(2) Suppose that $\overline{f}$ maps a convex open set $E \subseteq R^n$ into R^m, $\overline{f}$ is differentiable on E and there exists a constant M such that $\|f'\| \leq M$, $\forall\, x \in E$, then prove that

$|\overline{f}(b) - \overline{f}(a)| \leq M\,|b - a|$, $\forall\, a, b \in E$. (5)

b) (1) Suppose $\overline{f}$ maps an open set $E \subset R^n$ into R^m. Then prove that $\overline{f} \in \mathbf{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$ and $1 \leq j \leq n$.

OR

(2) If X is a complete metric space and if ϕ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\phi(x) = x$. (15)

III.

a)(1) Prove that (X), the set of all continuous, complex valued, bounded functions, defined on X, is a complete metric space with respect to the metric supremum norm.

OR

(2) If $\{f_n\}$ is a sequence of continuous functions defined on E and if $f_n \to f$ uniformly on E, then prove that f is continuous on E. (5)

b)(1) Let α be monotonically increasing function on [a,b]. Let $f_n \in \Re(\alpha)$ on [a,b], n = 1,2,.. . and let $f_n \to f$ uniformly on [a, b] then prove that $f \in \Re(\alpha)$ and

$$\int_a^b f\,d\alpha = \int_a^b f_n\,d\alpha \qquad (7)$$

(2) Suppose that $\{f_n\}$ is a sequence of differentiable functions on [a,b]. Suppose that $\{f_n(x_o)\}$ converges uniformly on [a,b] to some function f and then prove that

$$f'(x) = \lim_{x \to \infty} f_n'(x), a \leq x \leq b. \qquad (8)$$

OR

(3) State and prove Stone-Weierstrass theorm. (15)

IV.

a) (1)Suppose that the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and define f (x) = $\sum_{n=0}^{\infty} c_n x^n$, ($|x| < R$). Then prove that $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on [-R+∈, R-∈], no matter which ∈>0 is chosen. And prove that the function f is continuous and differentiable in (-R,R) and $f'(x) = \sum_{n=0}^{\infty} nC_n x^{n-1}$, ($|x| < R$).

OR

(2) Expand f(x) = x, $-\pi < x < \pi$, as a Fourier series with period 2π. (5)

b) (1) Suppose $\sum C_n$ converges. Put $f(x) = \sum_{n=0}^{\infty} c_n x^n$ (-1< x <1). Then prove that $\lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} c_n$. (7)

(2) State and prove Parseval's theorem. (8)

OR

(3) Explain with usual notations: Fourier series, orthogonal and orthonormal system. And prove the following theorem: Let $\{\phi_n\}$ be orthonormal on [a,b]. Let $S_n(x) = \sum_{m=1}^{n} c_m \phi_m(x)$ be the n^{th} partial sum of the Fourier series of f and suppose that $t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$. Then prove that $\int_a^b \left|f - S_n\right|^2 dx \le \int_a^b \left|f - t_n\right|^2 dx$ and equality holds if and only if $\gamma_m = c_m$, m = 1,2, …,n. (15)

V.

a) (1)Define Chebyshev polynomial and list down its properties.

OR

(2) If f has a derivative of order n at a point x_0, then prove that the Taylor Polynomial

$$P(x) = \sum_{k=0}^{n} \left(\frac{f^{(k)}(x_0)}{k!} \right) (x - x_0)^k$$ is the unique polynomial such that $\|f - P\| \le \|f - Q\|$

whatever Q may be in $P^{(n)}$. (5)

b)(1) Given n+1 distinct points $x_0, x_1, \ldots, x_n$ and n+1 real numbers $f(x_0)$, $f(x_1)$, …, $f(x_n)$ not necessarily distinct, then prove that there exists one and only one polynomial P of degree ≤ n such that $P(x_j) = f(x_j)$ for each j = 0,1,2,…,n. and the polynomial is given by the formula $P(x) = \sum_{k=0}^{n} \frac{f(x_k) A_k(x)}{A_k(x_k)}$ where $A_k(x) = \prod_{\substack{j=0 \\ i \ne k}}^{n} (x - x_j)$. (7)

(2) Let $P_{n+1}(x) = x^{n+1} Q(x)$ where Q is a polynomial of degree ≤ n and let $\|P_{n+1}\| =$ maximum of $|P_{n+1}(x)|$, a ≤ x ≤ b. Then prove that we get the inequality

$$\|P_{n+1}\| \ge \frac{\left(\frac{b-a}{2}\right)^{n+1}}{2^n}$$ with equality holding if and only if

$$P_{n+1} = \frac{(b-a)^{n+1}}{2^{2n+1}} T_{n+1}\left(\frac{2x - a - b}{b - a}\right).$$ (8)

OR

(3) Assume that the derivative $f^{(n+1)}$ exists on [a,b] and let T be the polynomial of degree ≤ n that best approximates f on [a,b] relative to the maximum norm. Then prove that there are (n+1) distinct points $x_0, x_1, \ldots, x_n$ in the open interval (a,b) such that for each x in [a,b] we get $f(x) - T(x) = \frac{A(x)}{(n+1)!} f^{(n+1)}(c)$ where $A(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$ and $\min\{x_0, x_1, \ldots, x_n, x\} < C < \max\{x_0, x_1, \ldots, x_n, x\}$. (15)

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