LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – NOVEMBER 2016

16PMT1MC02 - REAL ANALYSIS

Date: 04-11-2016 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

Answer all Questions. All questions carry equal marks.

1. (a) (i) Suppose f is a real function defined on R which satisfies $\lim_{h\to 0}[f(x+h)-f(x-h)]=0$, for every $x \in R$. Does this imply that f is continuous? **(5 marks)**

(OR)

(ii) Suppose f is a continuous mapping of a compact metric space X into a metric space. Then prove that $f(X)$ is compact. **(5 marks)**

(b) (i) Let A and B be disjoint nonempty closed subsets in a metric space X and define $f(p)=\frac{\rho_A(p)}{\rho_A(p)+\rho_B(p)}, p$ □ X, where $\rho_E(x)=inf_{z\in E}d(x,z)$. Show that f is a continuous function on X whose range lies in [0,1] and $f^{-1}(\{0\})=A$ and $f^{-1}(\{1\})=B$.

(9 marks)

(ii)Prove that for any monotonic function on (a,b), the set of points at which f is discontinuous is atmost countable. **(6 marks)**

(OR)

(c) (i) Suppose f is a continuous mapping of [0,1] into itself. Prove that f(x)=x for atleast one $x \in [0,1]$. **(8 marks)**

(ii) Assume that f is a continuous real function defined in (a,b) such that $f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$, $\forall x,y \in (a,b)$. Then prove that f is convex. **(7 marks)**

2. (a) (i) If f is continuous on $[a, b]$, then prove that $f \in$ □(α).

(OR)

(ii) If $f_1 \in \Re(\alpha)$ and $f_2 \in \Re(\alpha)$ on $[a, b]$, then prove that f_1+f_2□ □(α).

(b) (i) Define a refinement of a partition P. If P^* is a refinement of P then prove that $L(P,f,\propto) \le L(P^*,f,\propto)$ and $U(P^*,f,\propto) \le U(P,f,\propto)$. **(5 marks)**

(ii) State and prove a necessary condition and sufficient condition for a bounded real valued function to be a Riemann-Steiltjes integrable. **(10 marks)**

(OR)

(c) (i) Suppose □ increases on [a, b], $a \le x_0 \le b$, $\propto$ is continuous at x_0, $f(x_0)=1$ and f(x)=0 if $x \ne x_0$. Prove that $f \in$ □(α) and $\int_a^b f d\alpha = 0$. **(5 marks)**

(ii) Let $f \in$ □(α) on [a, b], $m \le f \le M$, φ be continuous on [m,M] and $h(x)=\varphi(f(x))$ on [a,b]. Then prove that $h \in$ □(α) **(10 marks)**

3. (a) (i) Prove that $\lim_{n\to\infty} f_n'(0) \neq f'(0)$ where $f_n(x) = \frac{\sin nx}{\sqrt{n}}, x\ real, n = 1,2 \ldots$

(OR)

(ii) Find for what values of x, the given series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges absolutely? **(5 marks)**

(b) (i) Prove that for $f_n(x) = \frac{x^2}{(1+x^2)^n}, x\ real,\ n = 0,1,2 \ldots$, the following:

1. $f_n(x)$ are continuous functions for any x and n.
2. $\sum_{n=0}^{\infty} f_n(x)$ is a convergent series and the limit of the sum is continuous.

(ii) If $\{f_n\}$ is a sequence of continuous functions on a set E and if $f_n \to f$ uniformly on E, then prove that f is continuous on E. **(5+ 10 marks)**

(OR)

(c) If $\{f_n\}$ is a sequence of differentiable functions on [a, b] such that $\{f_n(x_0)\}$ converges for $x_0 \in [a,b]$ and $\{f_n'\}$ converges uniformly on [a, b], then prove that $\{f_n\}$ converges uniformly on [a, b] to a function f and $\lim_{n\to\infty} f_n'(x) = f'(x)$. **(15 marks)**

4. (a) (i) State and prove the Bessel's Inequality and hence derive the Parseval's formula.

(OR)

(ii) Let $S = \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$, where $\varphi_0(x) = \frac{1}{\sqrt{2\pi}}, \varphi_{2n-1}(x) = \frac{\cos nx}{\sqrt{\pi}}$ and $\varphi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}}$, for n = 1, 2…. Prove that S is orthnormal on any interval of length 2 π. (5 marks)

(b) (i) State and prove Riesz-Fischer theorem.
(ii) State and prove Riemann-Lebesgue lemma. **(8+7 Marks)**

(OR)

(c) (i) Define Dirichlet's kernel and prove that $\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(2n+1)\frac{x}{2}}{2\sin\frac{x}{2}}, x \neq 2m\pi$

(ii) If $f \in L[0,2\pi]$, f is periodic with period 2π and $\{s_n\}$ is a sequence of partial sums of Fourier series generated by f, $s_n = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), n = 1,2 \ldots$ then prove that

$$s_n(x) = \frac{2}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n(t)dt$$ **(5+10 marks)**

5. (a) (i) If A, B,C $\in L(R^n, R^m)$ and c is a scalar then prove the following:

1. $\|A+B\| \leq \|A\| + \|B\|$
2. $\|cA\| = |c|\|A\|$
3. $\|A-C\| \leq \|A-B\| + \|B-C\|$.

(OR)

(ii) Suppose X is a complete metric space and ϕ is a contraction of X into X. Prove that there exist one and only one $x \in X$ such that $\phi(x) = x$. **(5 marks)**

(b) State and prove the inverse function theorem.

(OR)

(c) State and prove the implicit function theorem. **(15 marks)**

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