LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – NOVEMBER 2017

17/16PMT1MC01 - LINEAR ALGEBRA

Date: 02-11-2017 Dept. No. Max. : 100 Marks
Time: 01:00-04:00

Answer ALL the questions.

I. a) i) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Prove that the following statements are equivalent.

1. c is a characteristic value of T.
2. The operator $(T\text{-}cI)$ is singular.
3. det $(T\text{-}cI) = 0$.

(OR) (5)

ii) Let $A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$ be the matrix of a linear operator T defined on R^3 with respect to the standard ordered basis. Prove that A is diagonalizable.

b) i) Let T be a linear operator on a finite dimensional vector space V. Prove that the minimal polynomial for T divides the characteristic polynomial for T.

(OR) (15)

ii) Let V be a finite dimensional vector space over F and T be a linear operator on V. Then prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

II. a) i) Let T be a linear operator on an n-dimensional vector space V. Let A be an n x n matrix. Then prove that characteristic and minimal polynomials for T have the same roots, except for multiplicities.

(OR) (5)

ii) Let W be an invariant subspace for T. Then prove that the minimal polynomial for T_w divides the minimal polynomial for T.

b) i) State and prove Primary Decomposition theorem.

(OR) (15)

ii) Let T be a linear operator on a finite dimensional space V. If T is diagonalizable and if $c_1, \ldots, c_k$ are the distinct characteristic values of T, then prove that there exist linear operators $E_1, \ldots, E_k$ on V such that

(i) $T = c_1E_1 + \ldots + c_kE_k$.

(ii) $I = E_1 + \ldots + E_k$.

(iii) $E_iE_j = 0, i \neq j$.

(iv) Each E_i is a projection.

(v) The range of E_i is the characteristic space for T associated with c_i.

III. a) i) Let T be a linear operator on a vector space V and W a proper T-admissible subspace of V. Prove that W and cyclic subspace $Z(\alpha;T)$ are independent.

(OR) (5)

ii) If $\mathcal{B}$ is an ordered basis for W_i $1 \leq i \leq k$, then prove that the sequences $\mathcal{B} = (\mathcal{B}_1 \ldots, \mathcal{B}_k)$ is an ordered basis of W.

b) i) Let T be a linear operator on a finite-dimensional vector space V and let W_0 be a proper T-admissible subspace of V. There show that exist non-zero vectors $\sim_1,...,\sim_r$ in V with respective T-annihilators $p_1,..., p_r$ are such that

(i) $V = W_0 \oplus Z(\sim_1;T) \oplus ... \oplus Z(\sim_r;T)$;

(ii) $p_k \; divides \; p_k - 1, k = 2,...,r.$ (15)

(OR)

ii) Let T be a linear operator on a finite-dimensional vector space V. Let p and f be the minimal and characteristic polynomials for T, respectively. Then prove that

(i) p divides *f*.

(ii) p and f have the same prime factors, except for multiplicities.

(iii) *If* $p = f_1^{r_1} ... f_k^{r_k}$ is the prime factorization of p, then $f = f_1^{d_1} ... f_k^{d_k}$. where d_i is the nullity of $f_i(T)^{r_i}$ divided by the degree of f_i.

IV. a) i) For any linear operator T on a finite dimensional inner product space V then prove that there exists a unique linear operator T^* on V such that (Tα/β) = (α/T*β) for all α, β in V.

(OR) (5)

ii) Define the Quadratic form q associated with a symmetric bilinear form f and prove that

$$f(\sim, \quad) = \frac{1}{4}q(\sim + \quad) - \frac{1}{4}q(\sim - \quad).$$

b) i) Let *f* be a non-degenerate bilinear form on a finite dimensional vector space V. Prove that the set of all linear operators on V which preserve is a group under the operation composition. (8)

ii) State and prove Principal Axis Theorem. (7)

(OR)

i) Let V be a complex vector space and f be a bilinear form on V such that f(α, α) is real for every α. Then prove that f is Hermitian. (8)

ii) Let f be a form on a fdrs V and let A be a matrix of *f* in an ordered basis ***B***. Prove *f* is a positive form iff $A = A^*$ and all the principal minors of A are all positive. (7)

V. a. i) Define: Bilinear forms, Bilinear function, Skew Symmetric Bilinear form, Positive forms.

(OR) (5)

ii) Let F be the field of real numbers or the field of complex numbers. Let A be an nxn matrix over F. Show that the function g defined by $g(X, Y) = Y^*AX$ is a positive form on the space F^{nx1} if and only if there exists an invertible nxn matrix P with entries in F such that $A = P^*P$.

b) i) Let *V* be a finite dimensional vector space over the field of complex numbers. Let *f* be a symmetric bilinear form on *V* which has rank *r*. Then prove that there is an ordered basis $\mathcal{B} = \{ \;_1, \;_2, ... \;_n\}$ for *V* such that the matrix of *f* in the ordered basis ***B*** is diagonal and

$$f(\;_i, \;_j) = \begin{cases} 1, & \text{j=1,2,..r} \\ 0, & j > r \end{cases}.$$

(OR) (15)

ii) If *f* is a non-zero skew-symmetric bilinear form on a finite dimensional vector space *V* then prove that there exist a finite sequence of pairs of vectors, $(\sim_1, \;_1), (\sim_2, \;_2), ...(\sim_k, \;_k)$ with the following properties:

1) $f(\sim_j, \;_j)$=1, j=1,2,,…,k. 2) $f(\sim_i, \sim_j) = f(\;_i, \;_j) = f(\sim_i, \;_j) = 0, i \neq j$.

3) If W_j is the two dimensional subspace spanned by $\sim_j$ and $\;_j$, then V=$W_1 \oplus W_2 \oplus ... W_k \oplus W_0$ where W_0 is orthogonal to all $\sim_j$ and $\;_j$ and the restriction of *f* to W_0 is the zero form.

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